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Some Results on Transport Theory and Their Application to Monte Carlo Methods

JEROME SPANIER

Westinghouse Electric Corporation, Bettis Atomic Power Laboratory, West Mifflin, Pennsylvania

Submitted by Mario Juncosa

1. Let

$$\psi(P) = \int_\Gamma K(P, P')\,\psi(P')\,dP' + S(P) \tag{1}$$

be the integral equation for the directional neutron collision density $\psi(P) = \Sigma_t(P)\,\phi(P)$, where $\Sigma_t(P)$ is the total macroscopic cross section at P and $\phi(P)$ is the neutron flux (see, e.g., [1, Appendix 2] for a derivation and discussion of Eq. (1)). In Eq. (1), P denotes a generic point of euclidean phase space Γ and $S(P)$ is the density of first collisions, so normalized that $\int_\Gamma S(P)\,dP = 1$. It is implicit that $S(P) \geqslant 0$ and $K(P, P') \geqslant 0$.

In applying Monte Carlo methods to the estimation of integrals of the form

$$I = \int_\Gamma g(P)\,\psi(P)\,dP, \tag{2}$$

where g is some known bounded nonnegative function (usually a ratio of cross sections), it is desirable to establish a correspondence between the physical model, based on Eq. (1), and a probability model. The numerical calculation itself is based on the properties of the probability model in that the integral I is actually estimated by replacing it by estimates of the integral

$$I = \int_\Omega \xi\,d\mu, \tag{3}$$

where Ω is a sample space of all random walk histories in phase space, ξ is an unbiased estimator of I defined on Ω, and μ is a probability measure on Ω.

In [2] a discussion is presented of the construction of a probability measure μ on Ω based on the notion of a random walk process. This construction of a probability measure on an infinite product space using conditional probabilities on the factors is based on a rather technical theorem due to Tulcea [3]; we shall not present the details here. In this paper we prove a number of results which establish a firm correspondence between the physical model, based on the transport equation, and the probability model. When these results are specialized to the analog case, which is designed to mimic exactly the behavior of particles in the physical model, certain rather natural consequences obtain. In particular, we show that if the physical model is subcritical, then so is the probability model corresponding to this analog measure. Some of these results are undoubtedly familiar to workers in the field but the author has been unable to locate them in the readily accessible mathematical literature. We begin with some preliminary definitions and results.

2. A *random walk process* $\{f_n, p_n\}$ consists of a sequence $f_n(P_1, \ldots, P_n)$ of probability density functions defined on the product space Γ^n, together with a sequence $p_n(P_1, \ldots, P_n)$ of functions on Γ^n with the properties

$$F_n(P_1, \ldots, P_k, \infty, \ldots, \infty) \equiv F_k(P_1, \ldots, P_k), \qquad 1 \leqslant k < n, \tag{R1}$$

(the symbol ∞ denotes that point of Γ each of whose components is infinite), where

$$F_n(P_1, \ldots, P_n) = \int_{R(P_1, \ldots, P_n)} \cdots \int f_n(Q_1, \ldots, Q_n)\, dQ_1 \cdots dQ_n$$

is the distribution function of f_n, $R(P_1, \ldots, P_n)$ is the set of points $(t_1, \ldots, t_n) \in \Gamma^n$ satisfying $-\infty \leqslant t_1 \leqslant P_1, \ldots, -\infty \leqslant t_n \leqslant P_n$, and

$$0 \leqslant p_n(P_1, \ldots, P_n) \leqslant 1, \qquad n = 1, 2, \ldots \tag{R2}$$

for all $(P_1, \ldots, P_n) \in \Gamma^n$.

Intuitively, $f_n(P_1, \ldots, P_n)$ is the probability density of a random walk chain involving the ordered sequences of states $(P_1, \ldots, P_n)$ (to be thought of as collision points) and $p_n(P_1, \ldots, P_n)$ is the probability of terminating the chain at P_n. In most cases the random walk process will describe a Markov process although this is not a necessary restriction.

The space Ω of all random walk histories (essentially infinite sequences of points of Γ) has a natural decomposition

$$\Omega = \left(\bigcup_{k=1}^{\infty} \Lambda_k\right) \cup \Lambda_\infty, \tag{4}$$

where Λ_k consists of all chains which terminate after exactly k collisions and Λ_∞ consists of those chains which never terminate. A typical chain C of Λ_k would be written $C = (P_1, P_2, \ldots, P_k, P_k, P_k, \ldots)$ with $P_1 \neq P_2 \neq \cdots \neq P_k$. If μ denotes the measure constructed by Tulcea's theorem from the random walk process $\{f_n, p_n\}$, we shall say that the probability model is *subcritical* if $\mu(\Lambda_\infty) = 0$. This condition is certainly necessary to the construction of a practical computational model.

It follows from the construction of [2] that the sets Λ_k are measurable and, furthermore, if Λ is any measurable set and ξ is any random variable on Ω, the expected value of ξ taken over $\Lambda \cap \Lambda_k$ is

$$\int_{\Lambda\cap\Lambda_k} \xi \, d\mu = \int_\Gamma \cdots \int_\Gamma \chi_{\Lambda\cap\Lambda_k}(P_1, \dots, P_k)\, \xi(P_1, \dots, P_k) f_k(P_1, \dots, P_k)$$
$$\times \prod_{i=1}^{k-1} q_i(P_1, \dots, P_i)\, p_k(P_1, \dots, P_k)\, dP_1 \cdots dP_k , \qquad (5)$$

where $\chi_{\Lambda\cap\Lambda_k}$ is the characteristic function of the set $\Lambda \cap \Lambda_k$, $q_i = 1 - p_i$, and $\prod_{i=1}^{0} \cdots = 1$ by convention. In Eq. (5) and in later formulas we shall write $f(P_1, \dots, P_k)$ in place of $f(P_1, \dots, P_k, P_k, \dots)$ when $(P_1, \dots, P_k, P_k, \dots) \in \Lambda_k$ and f is a function on Ω. Eq. (5) is plausible since, in the integral, f_k gives the probability density for collisions at $P_1, \dots, P_k$, $\prod_{i=1}^{k-1} q_i$ guarantees that the chain does not terminate at any of the points $P_1, \dots, P_{k-1}$, p_k assures the termination at P_k, and $\chi_{\Lambda\cap\Lambda_k}$ restricts the integrand to points of $\Lambda \cap \Lambda_k$.

Now, since the Λ_k, $1 \leqslant k \leqslant \infty$, are measurable, so is Λ_∞ by virtue of Eq. (4) and the fact that the Λ_k, Λ_∞ are pairwise disjoint. We shall use Eqs. (4) and (5) to display the measure of Λ_∞.

THEOREM 1. *For any random walk process* $\{f_n, p_n\}$ *and corresponding measure* μ,

$$\mu(\Lambda_\infty) = \lim_{N\to\infty} \int_\Gamma \cdots \int_\Gamma f_N(P_1, \dots, P_N) \prod_{i=1}^{N} q_i(P_1, \dots, P_i)\, dP_1 \cdots dP_N . \qquad (6)$$

PROOF. We have

$$1 = \mu(\Omega) = \sum_{k=1}^{\infty} \mu(\Lambda_k) + \mu(\Lambda_\infty)$$

so that

$$\mu(\Lambda_\infty) = 1 - \sum_{k=1}^{\infty} \mu(\Lambda_k)$$
$$= 1 - \sum_{k=1}^{\infty} \int_\Gamma \cdots \int_\Gamma f_k(P_1, \dots, P_k) \prod_{i=1}^{k-1} q_i p_k \, dP_1 \cdots dP_k .$$

We first show that, for every N,

$$S_N \equiv \sum_{k=1}^{N} \int_\Gamma \cdots \int_\Gamma f_k \prod_{i=1}^{k-1} q_i p_k \, dP_1 \cdots dP_k = 1 - \int_\Gamma \cdots \int_\Gamma f_N \prod_{i=1}^{N} q_i \, dP_1 \cdots dP_N . \qquad (7)$$

Since $p_k = 1 - q_k$,

$$S_N = \sum_{k=1}^{N} \int_\Gamma \cdots \int_\Gamma f_k \prod_{i=1}^{k-1} q_i \, dP_1 \cdots dP_k - \sum_{k=1}^{N} \int_\Gamma \cdots \int_\Gamma f_k \prod_{i=1}^{k} q_i \, dP_1 \cdots dP_k$$

$$= \int_\Gamma f_1 \, dP_1 + \sum_{k=2}^{N} \int_\Gamma \cdots \int_\Gamma f_{k-1} \prod_{i=1}^{k-1} q_i \, dP_1 \cdots dP_{k-1}$$

$$- \sum_{k=1}^{N} \int_\Gamma \cdots \int_\Gamma f_k \prod_{i=1}^{k} q_i \, dP_1 \cdots dP_k$$

$$= 1 - \int_\Gamma \cdots \int_\Gamma f_N \prod_{i=1}^{N} q_i \, dP_1 \cdots dP_N, \qquad \text{proving (7).}$$

Making use of (R1) one can show that $S_N \leqslant S_{N+1}$ and since the sequence $\{S_N\}$ is bounded above by unity, it has a limit. Taking limits in (7) proves Theorem 1.

3. Based on the integral equation (1) for the collision density $\psi(P)$, the *analog random walk process* is defined by the choices

$$f_1(P_1) = S(P_1), \tag{8}$$

$$f_n(P_1, \ldots, P_n) = \left[\prod_{\ell=2}^{n} \frac{K(P_\ell, P_{\ell-1})}{c(P_{\ell-1})}\right] S(P_1), \qquad n \geqslant 2, \tag{9}$$

where

$$c(P) \equiv \int_\Gamma K(Q, P) \, dQ,$$

and

$$p_n(P_1, \ldots, P_n) = p(P_n) = \frac{\Sigma_a(P_n)}{\Sigma_t(P_n)}, \tag{10}$$

where Σ_a, Σ_t are, respectively, the absorption and total macroscopic cross sections. The quantity $c(P)$ represents the mean number of secondary particles per primary particle upon collision at P. In a nonmultiplying medium, one has

$$c(P) = 1 - p(P) = \frac{\Sigma_s(P)}{\Sigma_t(P)}$$

$$= q(P),$$

where Σ_s is the macroscopic scattering cross section, $\Sigma_s + \Sigma_a = \Sigma_t$. In a multiplying medium one has, in any event, the inequalities

$$q(P) \leqslant c(P) < \infty. \tag{11}$$

For the analog measure μ corresponding to the random walk process defined by Eqs. (8), (9), and (10), Theorem 1 gives

$$\mu(\Lambda_\infty) = \lim_{N\to\infty} \int_\Gamma \cdots \int_\Gamma \prod_{\ell=2}^{N} \frac{K(P_\ell, P_{\ell-1})}{c(P_{\ell-1})} S(P_1) \prod_{i=1}^{N} q(P_i)\, dP_1 \cdots dP_N$$

$$\leqslant \lim_{N\to\infty} \int_\Gamma \cdots \int_\Gamma \prod_{\ell=2}^{N} K(P_\ell, P_{\ell-1})\, S(P_1)\, dP_1 \cdots dP_N,$$

making use of (11) and (R2). For the case of a subcritical physical process (to be defined below) we shall show that the limit on the right side of inequality (12) vanishes, as well as the stronger result that the expected number of collisions is finite.

4. For nonanalog processes an additional condition is needed to assure that the probability model is subcritical. In most cases, to define nonanalog random walk processes $\{\hat{f}_n, \hat{p}_n\}$, one first defines $\hat{f}_n$ by formulas exactly like (8) and (9), but with S and K replaced by $\hat{S}$ and $\hat{K}$, which may be regarded as the source and kernel for an integral equation for a transformed collision density, $\hat{\psi}$:

$$\hat{\psi}(P) = \int_\Gamma \hat{K}(P, P')\, \hat{\psi}(P')\, dP' + \hat{S}(P). \tag{13}$$

As before we require that $\hat{S} \geqslant 0$, $\hat{K} \geqslant 0$, $\int_\Gamma \hat{S}(P)\, dP = 1$, and that

$$\sup_{P\in\Gamma} \hat{c}(P) = \sup_{P\in\Gamma} \int_\Gamma \hat{K}(Q, P)\, dQ = M < \infty.$$

The latter condition is necessary to ensure that the mean number of secondaries per primary is bounded in the transformed process. Mathematically, it means that if $u(P)$ is a function on Γ such that

$$\| u \|_1 \equiv \int_\Gamma | u(P) |\, dP < \infty, \tag{14}$$

then the integral operator $\hat{\mathscr{K}}$ defined by

$$\hat{\mathscr{K}} u(P) = v(P) = \int_\Gamma \hat{K}(P, P')\, u(P')\, dP'$$

satisfies $\| v \|_1 \leqslant M \| u \|_1$; that is, that $\hat{\mathscr{K}}$ is a bounded operator on the Banach space $L_1(\Gamma)$ consisting of all measurable functions u satisfying (14). To complete the definition $\{\hat{f}_n, \hat{p}_n\}$, one chooses $\hat{p}_n$ to be arbitrary except one requires the natural restriction $\hat{c}(P) = 0 \Leftrightarrow \hat{q}_j(P_1, \ldots, P_{j-1}, P) = 0$ for all $P_1, \ldots, P_{j-1}$.

The additional restriction that must be placed on the random walk process $\{\hat{f}_n, \hat{p}_n\}$ is that there exists an integer L such that for $n \geqslant L$,

$$\prod_{i=1}^{n} \frac{\hat{q}_i(P_1, ..., P_i)}{\hat{c}(P_i)} \leqslant 1 \tag{15}$$

for all $(P_1, ..., P_n) \in \Gamma^n$. This condition is used to establish an inequality like (12) for the measure $\hat{\mu}$. Thus, from Theorem 1 and the definition of $\{\hat{f}_n, \hat{p}_n\}$,

$$\hat{\mu}(\Lambda_\infty) = \lim_{N\to\infty} \int_\Gamma \cdots \int_\Gamma \prod_{\ell=2}^{N} \frac{\hat{K}(P_\ell, P_{\ell-1})}{\hat{c}(P_{\ell-1})} \hat{S}(P_1) \prod_{i=1}^{N} \hat{q}_i(P_1, ..., P_i)\, dP_1 \cdots dP_N$$

$$\leqslant \lim_{N\to\infty} \int_\Gamma \cdots \int_\Gamma \prod_{\ell=2}^{N} \hat{K}(P_\ell, P_{\ell-1})\, \hat{S}(P_1)\, dP_1 \cdots dP_N . \tag{16}$$

Henceforth we assume condition (15) for all nonanalog processes $\{\hat{f}_n, \hat{p}_n\}$.

In Eq. (13), the source $\hat{S}$ is used to construct a distribution of initial collision points and the kernel $\hat{K}$ is used to move particles from state to state. However, the particles may be thought of as carrying weights in order to adjust the expected weight undergoing collision at points of Γ to the analog value. Thus, a particle which suffers an initial collision at P_1 must be assigned a weight $S(P_1)/\hat{S}(P_1)$ and a particle which moves from a collision at P_i to one at P_{i+1} has its weight multiplied by the factor

$$\frac{K(P_{i+1}, P_i)}{\hat{K}(P_{i+1}, P_i)} \frac{\hat{c}(P_i)}{\hat{q}_i(P_1, ..., P_i)} .$$

Finally, when a particle history is terminated at P_k, its weight is multiplied by

$$\frac{p(P_k)}{\hat{p}_k(P_1, ..., P_k)} .$$

If we choose, at various stages of the calculation we may split a particle of weight W into independent fragments whose weights sum to W. This leads to a numerically different, though statistically equivalent procedure. If $W(P)$ denotes the expected weight density of particles undergoing collision at P, $W(P)$ may be shown to satisfy Eq. (1). In the subcritical case (see Theorem 3 below) this will enable us to identify $W(P)$ with $\psi(P)$ and to conclude that $\hat{\mu}(\Lambda_\infty) = 0$.

Mathematically, a consistency condition is needed in order to make the above process well defined in a measure-theoretic sense. The condition needed is that the analog measure μ be absolutely continuous with respect to the measure $\hat{\mu}$ constructed from the random walk process $\{\hat{f}_n, \hat{p}_n\}$, i.e., that sets of $\hat{\mu}$-measure zero also have μ-measure zero. Conditions which guarantee this are stated in

THEOREM 2. *Assume that*

$$\hat{S}(P) = 0 \Rightarrow S(P) = 0, \qquad \frac{\hat{K}(P,Q)}{\hat{c}(Q)} = 0 \Rightarrow \frac{K(P,Q)}{c(Q)} = 0,$$

$$\hat{p}_j(P_1, \ldots, P_{j-1}, P) = 0 \Rightarrow p(P) = 0, \qquad \hat{q}_j(P_1, \ldots, P_{j-1}, P) = 0 \Rightarrow q(P) = 0$$

and that

$$X(C) = X(P_1, \ldots, P_k)$$
$$= \prod_{j=2}^{k} \frac{K(P_j, P_{j-1})}{\hat{K}(P_j, P_{j-1})} \frac{q(P_{j-1})}{\hat{q}_{j-1}(P_1, \ldots, P_{j-1})} \frac{\hat{c}(P_{j-1})}{c(P_{j-1})} \frac{S(P_1)\, p(P_k)}{\hat{S}(P_1)\, \hat{p}_k(P_1, \ldots, P_k)},$$
$$C \in \Lambda_k, \tag{17}$$

is bounded except possibly for sets of $\hat{\mu}$-measure zero. Then μ is absolutely continuous with respect to $\hat{\mu}$.

PROOF. Suppose $\hat{\mu}(\Lambda) = 0$ for some measurable Λ. Then

$$0 = \hat{\mu}(\Lambda \cap \Lambda_k) = \int_\Gamma \cdots \int_\Gamma \chi_{\Lambda \cap \Lambda_k} \prod_{j=2}^{k} \left[\frac{\hat{K}(P_j, P_{j-1})}{\hat{c}(P_{j-1})} \hat{q}_{j-1}(P_1, \ldots, P_{j-1})\right]$$
$$\cdot \hat{S}(P_1)\, \hat{p}_k(P_1, \ldots, P_k)\, dP_1 \cdots dP_k$$

for all k, including $k = \infty$. Now consider

$$\mu(\Lambda \cap \Lambda_k) = \int_\Gamma \cdots \int_\Gamma \chi_{\Lambda \cap \Lambda_k} \prod_{j=2}^{k} \frac{K(P_j, P_{j-1})}{c(P_{j-1})} q(P_{j-1})\, S(P_1)\, p(P_k)\, dP_1 \cdots dP_k$$
$$= \int \cdots \int \chi_{\Lambda \cap \Lambda_k} X(P_1, \ldots, P_k) \prod_{j=2}^{k} \frac{\hat{K}(P_j, P_{j-1})}{\hat{c}(P_{j-1})} \hat{q}_{j-1}(P_1, \ldots, P_{j-1})$$
$$\times \hat{S}(P_1)\, \hat{p}_k(P_1, \ldots, P_k)\, dP_1 \cdots dP_k.$$

From this, one sees that

$$\mu(\Lambda \cap \Lambda_k) \leqslant C_1 \hat{\mu}(\Lambda \cap \Lambda_k) = 0,$$

where C_1 is the essential bound for X, so that

$$\mu(\Lambda) = \sum_{k=1}^{\infty} \mu(\Lambda \cap \Lambda_k) + \mu(\Lambda \cap \Lambda_\infty) = 0. \tag{18}$$

Equation (18) proves Theorem 2.

Theorem 2 implies that, when its hypotheses are satisfied, the Radon-Nikodym derivative $d\mu/d\hat{\mu}$ is defined up to $\hat{\mu}$-equivalence on Ω by $d\mu/d\hat{\mu} = X$. Then one may write [4, pp. 132-133].

$$\int_{\Omega} \xi \, d\mu = \int_{\Omega} \xi \frac{d\mu}{d\hat{\mu}} \, d\hat{\mu} \tag{19}$$

for all random variables ξ. Equation (19) states that if ξ is an unbiased estimator of the integral (2) with respect to the analog measure μ, then $\hat{\xi} = \xi(d\mu/d\hat{\mu})$ is an unbiased estimator of (2) with respect to the measure $\hat{\mu}$. The function $(d\mu/d\hat{\mu})$ (C) is the ratio of the weight of C in the analog process to the weight in the nonanalog process. The object of using nonanalog processes $\{\hat{f}_n, \hat{p}_n\}$ in place of the analog process is to reduce the second moment of $\hat{\xi}$ and thus to reduce the statistical uncertainty in the estimate of (2).

The hypotheses of Theorem 2 are overly restrictive in practice, however. What is often desired is that Eq. (19) hold, not necessarily for *all* random variables ξ, but only for certain ones used to estimate the integral (2). In order to satisfy this desire, one is often able to relax the conditions of Theorem 2. In particular, it is sometimes useful to allow $\hat{p}_j(P_1, \ldots, P_{j-1}, P) = 0$ with $p(P) \neq 0$, and this can usually be done without violating (19) for appropriate ξ. We shall not go into the details of this point.

We note that if we choose $S = \hat{S}$, $K = \hat{K}$, $p = \hat{p}$, the nonanalog process $\{\hat{f}_n, \hat{p}_n\}$ reduces to the analog process defined by Eqs. (8), (9), and (10). In this case we note that a weight factor $c(P)/q(P)$ is still needed upon collision at P, or else the total number of particles must be readjusted by splitting to account for $c(P)/q(P) > 1$. As we have pointed out earlier with Eq. (11), condition (15) is automatically satisfied for the analog process. With these facts in mind we shall prove a number of results in Section 5 about the general nonanalog process $\{\hat{f}_n, \hat{p}_n\}$, realizing that the analog process is a special case. In this sense, the results of Section 5 may be regarded as general results about the integral Eq. (13) with $\hat{S}$, $\hat{K} \geqslant 0$.

5. Returning to Eq. (13), we assume $\hat{S}$, $\hat{K} \geqslant 0$, $\int_{\Gamma} \hat{S}(P)\, dP = 1$, and that $\hat{\mathscr{K}}$ is a bounded operator on $L_1(\Gamma)$, as before.

As is usual, we shall use the spectrum of $\hat{\mathscr{K}}$, $\sigma(\hat{\mathscr{K}})$, to introduce the notion of the subcriticality of $\hat{\mathscr{K}}$. Accordingly, we recall [5, VII] that the resolvent set of $\hat{\mathscr{K}}$, $\rho(\hat{\mathscr{K}})$, is the set of complex numbers λ such that $\lambda I - \hat{\mathscr{K}}$ has a bounded inverse defined on $L_1(\Gamma)$ and $\sigma(\hat{\mathscr{K}})$ is the set of complex numbers not in $\rho(\hat{\mathscr{K}})$. We say that $\hat{\mathscr{K}}$ is *subcritical* if

$$\sigma(\hat{\mathscr{K}}) \subset \{\lambda : |\lambda| < 1\}. \tag{20}$$

THEOREM 3. *If $\hat{\mathscr{K}}$ is subcritical, then Eq. (13) has a unique solution $\hat{\psi}$ given by the Neumann series*

$$\hat{\psi} = \sum_{n=0}^{\infty} \hat{\mathscr{K}}^n \hat{S} \tag{21}$$

and the series converges in the L_1 norm.

PROOF. Since $1 \in \rho(\hat{\mathscr{K}})$, $\psi = (I - \hat{\mathscr{K}})^{-1} S$ is unique. Equation (21) is just the resolvent series [1, VII.3.4].

It is worth noting that the L_1-convergence of the Neumann series plays a major role in the proof of the unbiased nature of many estimators of Monte Carlo.

THEOREM 4. *If $\hat{\mathscr{K}}$ is subcritical and if condition (15) is satisfied, then $\hat{\mu}(\Lambda_\infty) = 0$.*

PROOF. The Nth term of the series (21) is the nonnegative function

$$\hat{\mathscr{K}}^N S(P) = \int_\Gamma \cdots \int_\Gamma \hat{K}(P, P_{N-1}) \cdots \hat{K}(P_2, P_1)\, \hat{S}(P_1)\, dP_1 \cdots dP_{N-1}$$

whose L_1 norm bounds $\hat{\mu}(\Lambda_\infty)$ (Eq. (16)) as N tends to infinity. By Theorem 3 this limit is zero.

It is sometimes useful to have the following criterion for subcriticality.

THEOREM 5. *If there is an $n > 0$ such that $\| \hat{\mathscr{K}}^n \|_1 < 1$, then $\hat{\mathscr{K}}$ is subcritical.*

PROOF. By using the spectral mapping theorem [1, VII.3.11] applied to the polynomial z^n,

$$\sigma(\hat{\mathscr{K}}^n) = \{\lambda^n : \lambda \in \sigma(\hat{\mathscr{K}})\}. \tag{22}$$

But [1, VII.3.4] $w \in \sigma(\hat{\mathscr{K}}^n)$ implies $| w | \leqslant \| \hat{\mathscr{K}}^n \|_1 < 1$. Hence from (22), $\lambda \in \sigma(\hat{\mathscr{K}})$ implies $| \lambda | = | w |^{1/n} < 1$ proving that $\hat{\mathscr{K}}$ is subcritical.

6. Theorem 4 is useful because the condition $\| \hat{\mathscr{K}}^n \|_1 < 1$ is often easier to verify than (20). For example, it is implied by the following condition on the kernel $\hat{\mathscr{K}}$:

(A) There exists a constant $c < 1$ and an integer N depending only on c such that the iterated integral

$$\int_\Gamma \cdots \int_\Gamma \hat{K}(P_N, P_{N-1}) \cdots \hat{K}(P_1, P)\, dP_1 \cdots dP_N \leqslant c < 1 \tag{23}$$

for all $P \in \Gamma$. As mentioned earlier, for the analog case in a nonmultiplying medium,

$$\int_\Gamma K(P, P')\, dP = \frac{\Sigma_s(P')}{\Sigma_t(P')} \leqslant 1 \qquad \text{for all} \qquad P' \in \Gamma,$$

where $\Sigma_s(P')$ is the macroscopic scattering cross section at P'. In such a case, if there is a subregion of Γ of positive measure for which $\Sigma_s(P) < \Sigma_t(P)$ (an absorbing subregion) and if the differential scattering cross section at all P is nonzero for all scattering directions, then it may be shown that

$$\int_\Gamma \int_\Gamma K(P_2, P_1)\, K(P_1, P)\, dP_1\, dP_2 < 1$$

for all P. Thus, the condition (A) is satisfied for $N = 2$. For points P at which certain scattering directions are prohibited, it may be necessary to take $N > 2$ to satisfy condition (A) but it should still be possible to satisfy the condition unless the only scattering permitted is δ-scattering (i.e., only a single exit direction permitted).

In multiplying media one must rely on sufficient absorption plus leakage to guarantee (A).

7. In Section 5 we established results which guarantee that the probability model be subcritical in the sense $\hat{\mu}(\Lambda_\infty) = 0$, under appropriate assumptions. It is also of interest to note that, using this result, a stronger result may be proved, namely that the expected weighted number of collisions is finite. We state this as follows:

THEOREM 6. *Define a random variable* $\hat{\xi}$ *by*

$$\hat{\xi}(C) = \sum_{i=1}^{k} \prod_{j=1}^{i-1} \frac{\hat{c}(P_j)}{\hat{q}_j(P_j, \ldots, P_j)}, \qquad C = (P_1, \ldots, P_k, P_k, \ldots) \in \Lambda_k .$$

and assume that $\hat{\xi}$ *is bounded except possibly on sets of* $\hat{\mu}$*-measure zero. Then* $E[\hat{\xi}] = \int_\Gamma \hat{\psi}(P)\, dP$ *if the conditions of Theorem 4 hold.*

PROOF. $E[\hat{\xi}]$ is certainly finite since $\hat{\xi}$ is bounded almost everywhere. Then

$$E[\hat{\xi}] = \sum_{k=1}^{\infty} \int_{\Lambda_k} \hat{\xi}\, d\hat{\mu}$$

$$= \sum_{k=1}^{\infty} \sum_{i=1}^{k} \int_\Gamma \cdots \int_\Gamma \prod_{j=1}^{i-1} \frac{\hat{c}(P_j)}{\hat{q}_j} \prod_{j=2}^{k} \frac{\hat{K}(P_j, P_{j-1})}{\hat{c}(P_{j-1})}\, \hat{S}(P_1) \prod_{j=1}^{k-1} \hat{q}_j \hat{p}_k\, dP_1 \cdots dP_k .$$

Permute the double sum $\sum_{k=1}^{\infty}\sum_{i=1}^{k}$ to $\sum_{i=1}^{\infty}\sum_{k=i}^{\infty}$ to get

$$E[\hat{\xi}] = \sum_{i=1}^{\infty} \int_{\Gamma} \cdots \int_{\Gamma} \prod_{j=1}^{i-1} \frac{\hat{c}(P_j)}{\hat{q}_j} \prod_{j=2}^{i} \frac{\hat{K}(P_j, P_{j-1})}{\hat{c}(P_{j-1})} \hat{S}(P_1) \prod_{j=1}^{i-1} \hat{q}_j$$

$$\times \left\{ \hat{p}_i(P_i) + \int_{\Gamma} \hat{K}(P, P_i) \frac{\hat{q}_i}{\hat{c}(P_i)} \hat{p}_{i+1}(P)\, dP + \cdots \right\} dP_1 \cdots dP_i .$$

By using $\hat{p} = 1 - \hat{q}$ and $\hat{\mu}(\Lambda_\infty) = 0$, the term in braces may be shown to collapse to 1. Thus

$$E[\hat{\xi}] = \sum_{i=1}^{\infty} \int_{\Gamma} \cdots \int_{\Gamma} \prod_{j=2}^{i} \hat{K}(P_j, P_{j-1})\, \hat{S}(P_1)\, dP_1 \cdots dP_i = \int_{\Gamma} \hat{\psi}(P)\, dP$$

by Theorem 3.

Note that if we specialize to a case where $\hat{c} = \hat{q}$ (nonmultiplying), then $\hat{\xi}(C) = k$ as one would expect, and $E[\hat{\xi}]$ is simply the expected number of collisions. The finiteness of this quantity automatically shows that $\hat{\mu}(\Lambda_\infty) = 0$. This will also follow for any case in which $\hat{c} \geqslant \hat{q}$, as in the analog case.

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